Hindsight:
LSCM and DNCP are one and the same

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Goal

In this short note, we show the equivalence of the Least-Square Conformal Maps [4] (LSCM) and the Discrete, Natural Conformal Parameterization [1] (DNCP).

Conformal Energy vs. Dirichlet Energy

As mentioned in [5], the conformal energy E_C and the Dirichlet energy E_D of a mapping f (between a surface patch S and a (u,v) region) are linked , since:

$$E_C(f) = E_D(f) - A(f)$$

where E_C measures the failure for the map f to be conformal, E_D is the integral of the square of the magnitude of the gradient of f, and $A(f)$ is the surface area of the (u,v) region. **A conformal map therefore corresponds to a map f that is a critical point of both E_C and E_D.**

LSCM = DNCP

Previous methods have already exploited this variational definition of a conformal map using, equivalently, either E_C and E_D to find conformal maps for constrained boundary vertices (compare, for instance, [3] and [2]).

However, [4] and [1] proposes a minimally constrained technique, where only 2 vertices are pinned down. Since these techniques minimize respectively E_C and E_D for the interior of the patch, the equivalence is straightforward. The boundary condition in [1] is, however, not obviously equivalent to the other approach.

It turns out that this Neumann boundary condition corresponds to the conformality condition. Indeed, the boundary condition can be straightforwardly rewritten as $\nabla E_D = \nabla A$ (the lhs leads to the cotangent formula, while the gradient of the area is expressed on the rhs using only the 1-ring vertices). Therefore, we get: $\nabla(E_D - A) = \nabla(E_C) = 0$. In other words, solving the linear system in [1] is equivalent to solving the linear system established in [4]: they will both lead to the same critical point of both energies E_C and E_D. Which also disproves the claim of no-foldover guarantee made in [4]. Note, however, that the equivalence is only theoretical: the numerics used to find the critical point are different, and their respective efficiency may highly depend on the preconditioning and solver used.

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References

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