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On the geometric character of stress in continuum mechanics

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Abstract. This paper shows that the stress field in the classical theory of continuum mechanics may be taken to be a covector-valued differential two-form. The balance laws and other fundamental laws of continuum mechanics may be neatly rewritten in terms of this geometric stress. A geometrically attractive and covariant derivation of the balance laws from the principle of energy balance in terms of this stress is presented.

Mathematics Subject Classification (2000).

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1. Motivation

This paper proposes a reformulation of classical continuum mechanics in terms of bundle-valued exterior forms. Our motivation is to provide a geometric description of force in continuum mechanics, which leads to an elegant geometric theory and, at the same time, may enable the development of space-time integration algorithms that respect the underlying geometric structure at the discrete level.

In classical mechanics the traditional approach is to define all the kinematic and kinetic quantities using vector and tensor fields. For example, velocity and traction are both viewed as vector fields and power is defined as their inner product, which is induced from an appropriately defined Riemannian metric. On the other hand, it has long been appreciated in geometric mechanics that *force* should not be viewed as a vector, but rather a one-form. This fits naturally with one of the main properties of a force, namely that when paired with a displacement (a vector), one gets *work*. No metric is needed for this operation of course when force is thought of as a one form. One also sees the same thing when one looks at the tensorial nature of the Euler–Lagrange equations: the equations themselves are natually one-form equations, not vector equations. Despite this, the notion of force as a one-form has not properly been put into the foundations of continuum mechanics. In the

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geometric approach to continuum mechanics proposed in this paper, traction is defined as an exterior one-form. Consequently, one also has a *metric-independent* notion of *power* as the natural pairing between the velocity vector field and the traction one-form.

Although the importance of the geometric character of these fields is already known in mechanics (see, for example, [17] and [3]), the classical derivation of the balance laws as presented in most works does not reflect this geometric understanding. *One of the purposes of the present work is to fill this gap.*

An outcome of this approach is that the stress field is naturally described as a bundle-valued two-form. The balance laws are then rewritten in terms of the new geometric stress by appealing to tools from differential calculus on bundle-valued forms; that is, in terms of Cartan's calculus [2]. It is worth noting that the notion of stress as a covector-valued two-form appears in the recent literature, e.g., [18, 9, 13, 6], but a reformulation of the balance laws in terms of this stress in arbitrary Riemannian ambient spaces has remained open. This paper fills that gap and provides a complete treatment of continuum mechanics, including balance laws and constitutive equations, in terms of this geometric notion of the stress.

The reformulation of elasticity in terms of (bundle-valued) exterior forms brings the theory closer to Discrete Exterior Calculus (DEC), see, e.g., [14] and [5], and, therefore, may contribute to the development of discrete mechanics and structure-preserving integration schemes. The systematic design of algorithms that preserve exactly the conservation laws of momentum and energy or exhibit dissipation consistent with the continuous systems (no spurious numerical dissipation) for any step-size is an active area of research; see, for example, the work on symplectic and variational integrators in [15], [19], [20], [22] and references therein. While such time integrators for finite-dimensional mechanical systems are well-understood, space-time integration algorithms that respect the geometric character of the physical quantities (such as stress and strain) and the symmetries of the equations (such as conservation of momentum and energy) remain a challenge. Discrete Exterior Calculus alone might not, in itself, be sufficient for the design of such conserving algorithms but may provide some useful tools for this undertaking. We view the present study, that is, the geometric reformulation of elasticity, as a first step in our research project on developing a consistent theory of discrete elasticity that will lead to the design of geometric space-time integration algorithms.

The organization of this paper is as follows: In §2 we introduce the stress as a bundle-valued form and rewrite the classical balance laws and constitutive relations in terms of this geometric stress. In §3, we assume the existence of a stress form, with no reference to the stress tensor, and present a covariant derivation of the balance laws and constitutive equations. The results are summarized in §4.

2. Classical continuum mechanics in terms of bundle-valued forms

Bodies and motions. As in traditional continuum mechanics, a *body* is a set of particles, or material points, which are often regarded as a subset of Euclidean 3-space. Following the geometric view of continuum mechanics[1], in this paper we shall regard a body as a Riemannian manifold $\mathcal{R}_0$ with boundary, whose metric tensor is denoted $\mathbf{G}$. Of course a standard example would be an open set with a sufficiently smooth boundary in Euclidean 3-space $\mathbb{R}^3$ for three-dimensional elasticity and a domain in $\mathbb{R}^2$ for shells, each with the standard metric. The body is assumed to deform in an ambient Riemannian space $\boldsymbol{\mathcal{S}}$ with metric $\mathbf{g}$ (Euclidean 3-space in the standard examples). A *deformation* is a map $\varphi : \mathcal{R}_0 \to \boldsymbol{\mathcal{S}}$ that is a diffeomorphism from $\mathcal{R}_0$ onto its image, the deformed body $\mathcal{R} = \varphi(\mathcal{R}_0) \subset \boldsymbol{\mathcal{S}}$. The deformed body inherits the Riemannian structure of $\boldsymbol{\mathcal{S}}$; see Figure 1. A *motion* is a curve of deformations, that is, a one-parameter (time dependent) family of deformation maps and may be represented as

$$\mathbf{x} \;=\; \varphi_t(\mathbf{X}) \;=\; \varphi(\mathbf{X}, t) \;, \tag{1}$$

where $\mathbf{X}$ and $\mathbf{x}$ denote, respectively, the position of a material particle in the fixed reference configuration $\mathcal{R}_0$ and its position in the current configuration $\mathcal{R}$.

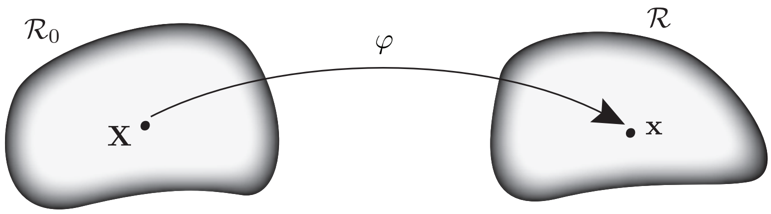

Figure 1. The deformation map φ takes the reference configuration $\mathcal{R}_0$ to the current configuration $\mathcal{R}$.

Further background and notation. The tangent bundle of $\mathcal{R}$ is denoted by $T\mathcal{R}$ and the cotangent bundle by $T^*\mathcal{R}$. Elements of $T\mathcal{R}$ are called *vectors* and those of $T^*\mathcal{R}$ are *covectors* or *one-forms*. Consider a coordinate basis $\mathbf{e}_i = \partial/\partial x^i$, $i = 1, 2, 3$ for $T\mathcal{R}$ and its dual basis $\mathbf{e}^i = \mathrm{d}x^i$ for $T^*\mathcal{R}$, where x^i represents the local coordinates of a point $\mathbf{x}$ in $\mathcal{R}$. Recall that a covariant, second-order tensor field (or 2-tensor) $\mathbf{T}$ on $\mathcal{R}$ is a bilinear mapping $\mathbf{T} : T\mathcal{R} \times T\mathcal{R} \to \mathbb{R}$ and can be expressed in coordinates as $\mathbf{T} = T_{ij}\mathbf{e}^i \otimes \mathbf{e}^j$, where $\otimes$ denotes the tensor product. A contravariant 2-tensor $\mathbf{S}$ is a bilinear mapping $\mathbf{S} : T^*\mathcal{R} \times T^*\mathcal{R} \to \mathbb{R}$, that can be expressed as $\mathbf{S} = S^{ij}\mathbf{e}_i \otimes \mathbf{e}_j$. One can also define mixed 2-tensors as bilinear

[1] It is assumed that the reader is familiar with the geometric approach to continuum mechanics. Further background and details on the notation can be found in [1] and [17] as well as in the research literature, such as [23].

mappings $T\mathcal{R} \times T^*\mathcal{R} \to \mathbb{R}$ or $T^*\mathcal{R} \times T\mathcal{R} \to \mathbb{R}$. We denote the space of covariant 2-tensor fields on $\mathcal{R}$ by $T_2^0(\mathcal{R})$, the space of contravariant 2-tensors by $T_0^2(\mathcal{R})$ and mixed tensors by $T_1{}^1(\mathcal{R})$ or $T^1{}_1(\mathcal{R})$. This notation extends in the obvious way to k-tensors. Further, let $\Omega^k(\mathcal{R})$ denote the space of k-forms, or alternating k-tensors, on $\mathcal{R}$. In particular, $\Omega^0(\mathcal{R})$ is the space of smooth functions on $\mathcal{R}$ and $\Omega^1(\mathcal{R})$ is the space of smooth sections of $T^*\mathcal{R}$. Similarly, we introduce a basis $\mathbf{E}_i$ on $T\mathcal{R}_0$ and its dual $\mathbf{E}^i$ on $T^*\mathcal{R}_0$ and adopt analogous notation for tensors on $\mathcal{R}_0$ as well as for two-point tensors, that is, tensors that can have "legs" in both $\mathcal{R}$ and $\mathcal{R}_0$ connected through the diffeormorphism $\varphi : \mathcal{R}_0 \to \mathcal{R}$. For example, $T^{0,0}_{1,1}(\mathcal{R}, \mathcal{R}_0)$ denotes the space of two-point 2-tensors or bilinear maps $T\mathcal{R} \times T\mathcal{R}_0 \to \mathbb{R}$.

Finally, the flat $(\cdot)^\flat$ and sharp $(\cdot)^\sharp$ operations refer to lowering and raising tensor indices. On $\mathcal{R}$ this would mean using the metric $\mathbf{g}$. For example, $^\flat : T\mathcal{R} \to T^*\mathcal{R}$ is defined by $\langle v^\flat, w\rangle = \mathbf{g}(v, w)$ and its inverse is $^\sharp : T^*\mathcal{R} \to T\mathcal{R}$. Similar operations are defined on the reference configuration $\mathcal{R}_0$ with respect to a metric $\mathbf{G}$. Here, the symbol $\langle\cdot,\cdot\rangle$ is used to denote the natural pairing of a contravariant field with a covariant field, such as the pairing of a vector field and a one-form or covector. We shall sometimes use the notation $\langle\langle\cdot,\cdot\rangle\rangle$ to denote the inner product between two covariant or contravariant fields with respect to the corresponding metric.

Continuum Mechanics. The motion (1) is assumed to occur due to the action of body forces per unit mass and surface traction forces per unit area of the boundary $\partial\mathcal{R}$. Continuum mechanics aims at providing the dynamical equations governing the motion under these conditions. In this paper, we adopt the standpoint that forces are one-forms as explained in §1, hence, the surface traction $\mathbf{t}$ and the body force $\mathbf{b}$ are naturally defined as one-forms and represented as vector fields through the $\sharp$ operator. This view is important to the development of Elasticity in terms of bundle-valued forms.

2.1. The stress field as a covector-valued two-form

Surface traction and Cauchy's stress. In the classical non-relativistic theory, the basic postulate for formulating the dynamical equations of motion is the existence of a stress field $\mathbf{t}^\sharp(\mathbf{x}, t; \mathbf{n})$ defined everywhere in $\mathcal{R}$. Physically, $\mathbf{t}^\sharp(\mathbf{x}, t; \mathbf{n})$ represents the force per unit area exerted on a surface element $\mathrm{d}a$ in $\mathcal{R}$ oriented with unit normal $\mathbf{n}$. It is also convenient to introduce the stress field $\mathbf{p}^\sharp(\mathbf{X}, t; \mathbf{N})$ acting on surface elements in $\mathcal{R}$ but measured per unit area of the corresponding surface elements in $\mathcal{R}_0$, that is,

$$\mathbf{p}^\sharp(\mathbf{X}, t; \mathbf{N})\, \mathrm{d}A \;=\; \mathbf{t}^\sharp\big(\mathbf{x}(\mathbf{X}, t), t; \mathbf{n}\big)\, \mathrm{d}a, \tag{2}$$

where $\mathrm{d}A$ is an oriented surface element in $\mathcal{R}_0$ with unit normal $\mathbf{N}$ and $\mathbf{N}\mathrm{d}A$ is related to $\mathbf{n}\mathrm{d}a$ by the Piola formula: $\mathbf{n}\mathrm{d}a = J\mathbf{F}^{-\mathsf{T}}\mathbf{N}\mathrm{d}A$. Here, the deformation gradient[2] is denoted $\mathbf{F} = \partial\varphi/\partial\mathbf{X}$ ($= F^a_A \mathbf{e}_a \otimes \mathbf{E}^A$) is a mixed 2-tensor $\in T^{1,0}_{0,1}(\mathcal{R}, \mathcal{R}_0)$,

[2] We cannot resist making the standard remark that despite its misleading name, $\mathbf{F}$ is not a

$J = \det(\mathbf{F})\sqrt{\det(\mathbf{g})/\det(\mathbf{G})}$ (det is the determinant), and $(\cdot)^{\mathsf{T}}$ denotes the transpose.

Cauchy's stress theorem states that there are second-order stress tensors called, respectively, the **Cauchy stress tensor $\boldsymbol{\sigma}$** and the **two-point Piola-Kirchhoff stress tensor $\mathbf{P}$**, such that

$$\mathbf{t}(\mathbf{x}, t; \mathbf{n}) \ = \ \langle \boldsymbol{\sigma}(\mathbf{x}, t), \, \mathbf{n} \rangle \, , \qquad \mathbf{p}(\mathbf{X}, t; \mathbf{N}) \ = \ \langle \mathbf{P}(\mathbf{X}, t), \, \mathbf{N} \rangle \ . \tag{3}$$

This means that $\mathbf{t}(\mathbf{x}, t; \mathbf{n})$ and $\mathbf{p}(\mathbf{X}, t; \mathbf{N})$ depend linearly on $\mathbf{n}$ and $\mathbf{N}$, respectively.

Rewriting the stress as a covector-valued two-form. Although the physical interpretation of the notion of stress is geometric, their vectorial and tensorial representations fail to exploit, or even reveal, their geometric character. One of the main goals of the present work is to clarify this geometric nature by rewriting the stress fields as covector-valued two-forms.

We will take an approach to stress that considers them to be covector valued two-forms and regards them as fundamental quantities in a manner similar to the way one postulates the existence of $\mathbf{t}(\mathbf{x}, t; \mathbf{n})$ in the standard approach. However, before taking this point of view, we show how they will end up being related to the standard notions. Namely, if we imagine the standard quantities being given, we define the "new" stresses $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{P}}$ in terms of them by applying the Hodge star operation $*_2$ to the second 'leg' of $\boldsymbol{\sigma}$ and $\mathbf{P}$ respectively as follows:

$$\boldsymbol{\mathcal{T}} = *_2\boldsymbol{\sigma}, \qquad \boldsymbol{\mathcal{P}} = *_2\mathbf{P}. \tag{4}$$

That is, in coordinate notation, one has: $\boldsymbol{\mathcal{T}} = \sigma_{ab} \ \mathbf{e}^a \otimes (*\mathbf{e}^b)$, and $\boldsymbol{\mathcal{P}} = P_{aA} \ \mathbf{e}^a \otimes (*\mathbf{E}^A)$.

By definition, one obtains $\boldsymbol{\mathcal{T}} \in \Omega^1(\mathcal{R}) \otimes \Omega^2(\mathcal{R})$ and $\boldsymbol{\mathcal{P}} \in \Omega^1(\mathcal{R}) \otimes \Omega^2(\mathcal{R}_0)$.[3] Physically, $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{P}}$ can be interpreted as follows: *the stress, upon pairing with a velocity field, provides an area-form that is ready to be integrated over a surface to give the rate of work done by the stress on that surface* — this point is elaborated further in §2.2.

Another point is worth mentioning. That is, part of the linearity of $\mathbf{t}$ is that it switches sign under a change of sign of $\mathbf{n}$. This property is nicely built into the new tensors $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{P}}$ simply because they are two forms—changing the arguments as two-forms switches their signs and this may be regarded as a reflection of the change of orientation of the surface to which $\mathbf{n}$ is normal. However, note that when we say $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{P}}$ are covector valued two forms, there is no need to mention $\mathbf{n}$ or a surface element da as such—unless one wants to reconstitute the classical stresses from them using oriented surface elements.

The Piola transformation. Recall that the standard stress tensors $\boldsymbol{\sigma}$ and $\mathbf{P}$ are related through the Piola transformation:

$$J\boldsymbol{\sigma} \ = \ \mathbf{P}\mathbf{F}^T. \tag{5}$$

gradient at all, but simply is the derivative of the map φ.

[3] Or, $\boldsymbol{\mathcal{T}} \in T\mathcal{R} \otimes \Omega^2(\mathcal{R})$ and $\boldsymbol{\mathcal{P}} \in T\mathcal{R} \otimes \Omega^2(\mathcal{R}_0)$, depending on the representation of $\boldsymbol{\sigma}$ and $\mathbf{P}$.

Clearly, $\mathbf{P}$ is not the pull-back of $\boldsymbol{\sigma}$ by the motion φ. But this equation, when written in terms of the stress-forms $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{P}}$, reads as:

$$\boldsymbol{\mathcal{P}} = \varphi^{*_2}\boldsymbol{\mathcal{T}}, \tag{6}$$

where φ^{*_2} is defined as the pull back by the mapping φ of the area-form of a covector-valued two-form, e.g., $\boldsymbol{\mathcal{T}} \in T^*(\mathcal{R}) \times \Omega^2(\mathcal{R})$, or of the second 'leg' of a two-tensor, e.g., $\boldsymbol{\sigma} \in T^*(\mathcal{R}) \times T^*(\mathcal{R})$. That is, the Piola transformation in (6) has a clear geometric interpretation: *$\boldsymbol{\mathcal{P}}$ is the pull-back of the area-form part of $\boldsymbol{\mathcal{T}}$ that does nothing to the covector-valued part.*

2.2. Physical interpretation of the stress form

Rate of work done by the stress. The rate of work $R^{\mathbf{t}}$ done by the traction forces $\mathbf{t}$ on an oriented surface S of the continuum can be written as

$$R^{\mathbf{t}} = \int_S \langle \mathbf{v}, \mathbf{t}\rangle \, \mathrm{d}a = \int_S \langle \mathbf{v}, \boldsymbol{\sigma}(\cdot, \mathbf{n})\rangle \, \mathrm{d}a = \int_S \boldsymbol{\sigma}(\mathbf{v}, \mathbf{n}) \, \mathrm{d}a = \int_S \langle \boldsymbol{\sigma}(\mathbf{v}, \cdot), \mathbf{n} \, \mathrm{d}a\rangle \tag{7}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ is the spatial velocity field. Surface integrals (over oriented surfaces) are more naturally expressed in terms of two-forms (that replace $\mathbf{n}\mathrm{d}a$). To this end, one can readily check that

$$R^{\mathbf{t}} = \int_S \langle \boldsymbol{\sigma}(\mathbf{v}, \cdot), \mathbf{n} \, \mathrm{d}a\rangle = \int_S *_2\boldsymbol{\sigma}(\mathbf{v}, \cdot) = \int_S \langle \mathbf{v}, *_2\boldsymbol{\sigma}\rangle = \int_S \langle \mathbf{v}, \boldsymbol{\mathcal{T}}\rangle \tag{8}$$

The above equation reads naturally as follows: *the rate of work done by the stress on an oriented hypersurface S is obtained by pairing the stress $\boldsymbol{\mathcal{T}}$ with the velocity field and integrating the resulting area-form over S.* Notice that if the orientation of S switches, then the sign of the integral automatically switches and this corresponds to the change of sign of $\mathbf{n}$ in the traditional approach. Similar relations hold for $\boldsymbol{\mathcal{P}}$ and $\mathbf{p}$, namely, $R^{\mathbf{t}} = \int_{S_0} \langle \mathbf{V}, \boldsymbol{\mathcal{P}}\rangle$, where $\mathbf{V}$ is the material velocity field defined by $\mathbf{V}\big(\mathbf{X}, t\big) = \partial\varphi(\mathbf{X}, t)/\partial t$ and $\mathbf{v}(\mathbf{x}, t) = \mathbf{V}\big(\mathbf{X}(\mathbf{x}, t), t\big)$.

The resultant force in Euclidean space. Let the body deform in a Euclidean space. The notion of a resultant force acting on a surface S with unit normal $\mathbf{n}$ depends on the Euclidean structure of the ambient space and is given by $\mathbf{f} = \int_S \mathbf{t}\mathrm{d}a = \int_S \langle \boldsymbol{\sigma}, \mathbf{n}\mathrm{d}a\rangle$. This force can be rewritten as

$$\mathbf{f} = \int_S \boldsymbol{\mathcal{T}} = \int_S *_2\boldsymbol{\sigma} = \int_S \mathbf{e}^a \otimes *(\sigma_{ab}\mathbf{e}^b) = \int_S \mathbf{e}^a \otimes *\mathbf{t}_a = \mathbf{e}^a \otimes \int_S *\mathbf{t}_a, \tag{9}$$

where the Euclidean structure allows us to "factor out" the basis vectors of the traction field and integrate the area-form component-wise. The force in the latter expression does not explicitly depend on the normal $\mathbf{n}\mathrm{d}a$. Also, it is clear that $\boldsymbol{\mathcal{T}}$ automatically obeys **Cauchy's lemma** in the sense that the resultant force changes sign if we change the orientation of S, as we have mentioned previously.

2.3. Differentation of bundle-valued forms

In this section, we define an operation that will be of importance for rewriting the balance laws in terms of $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{P}}$ in §2.4, namely, a differentiation operation $\mathfrak{d}$ of vector- and covector-valued forms. The differentiation $\mathfrak{d}$ combines the exterior derivative $\mathbf{d}$, that has a topological character, with the covariant derivative ∇ with respect to the Riemannian connection, that has a metric character, see, e.g., [1] and [8]. To this end, recall that, in component notation, the covariant derivative $\nabla \boldsymbol{v}$ of a vector field $\boldsymbol{v} = v^i \mathbf{e}_i$ on $T\mathcal{R}$ is given by $\nabla_j v^i = v^i{}_{|j} = \partial v^i / \partial x^j + \gamma^i_{jk} v^k$, where γ^i_{jk} are the Christoffel symbols, also called the connection coefficients. This suggests that $\nabla \boldsymbol{v}$ can be expressed as a mixed 2-tensor, that is, a vector-valued one-form $\nabla \boldsymbol{v} = v^i{}_{|j} \mathbf{e}_i \otimes \mathbf{e}^j$. In particular, one has $\nabla \mathbf{e}_j = \mathbf{e}_i \otimes \gamma^i_{jk} \mathbf{e}^k = \mathbf{e}_i \otimes \omega^i_j$, where $\omega^i_j = \gamma^i_{jk} \mathbf{e}^k$ are called the connection one-forms.

The derivative $\mathfrak{d}$. Let $\mathbb{T}$ denote either $T\mathcal{R}$ or $T^*\mathcal{R}$, and let k be any integer ≤ 3. We define the differential operator

$$\mathfrak{d} : \mathbb{T} \otimes \Omega^{k-1}(\mathcal{R}) \longrightarrow \mathbb{T} \otimes \Omega^k(\mathcal{R}); \quad \boldsymbol{\mathcal{T}} \longmapsto \mathfrak{d}\boldsymbol{\mathcal{T}}$$

by

$$\langle \boldsymbol{u}, \mathfrak{d}\boldsymbol{\mathcal{T}} \rangle = \mathbf{d}(\langle \boldsymbol{u}, \boldsymbol{\mathcal{T}} \rangle) - \nabla \boldsymbol{u} \dot{\wedge} \boldsymbol{\mathcal{T}}, \tag{10}$$

for all $\boldsymbol{u} \in \mathbb{T}^*$, where $\dot{\wedge}$ is, by definition, an inner product or a pairing on the first 'leg' and a wedge product on the second 'leg'. For example, if one considers $\mathbf{T} = \mathbf{a} \otimes \mathbf{b}$ and $\mathbf{S} = \mathbf{c} \otimes \mathbf{d}$ both in $T(\mathcal{R}) \otimes \Omega^1(\mathcal{R})$, one gets $\mathbf{T} \dot{\wedge} \mathbf{S} = \langle\!\langle \mathbf{a}, \mathbf{c} \rangle\!\rangle \mathbf{b} \wedge \mathbf{d}$ in $\Omega^2(\mathcal{R})$.[4] Note that for $k = 0$, $\mathfrak{d}$ reduces to the regular covariant derivative, while for $k = 3$, $\mathfrak{d}$ is identically zero.

Now, in order for (10) to provide a valid definition of $\mathfrak{d}$, one needs to show that its right hand side depends only on the point values of $\boldsymbol{u}$ and, hence, uniquely defines the differential $\mathfrak{d}\boldsymbol{\mathcal{T}}$. To this end, note that for any function $f \in \Omega^0(\mathcal{R})$, one has

$$\mathbf{d}(\langle f\, \boldsymbol{u}, \boldsymbol{\mathcal{T}} \rangle) = \mathbf{d}(f \wedge \langle \boldsymbol{u}, \boldsymbol{\mathcal{T}} \rangle) = (\mathbf{d}f) \wedge \langle \boldsymbol{u}, \boldsymbol{\mathcal{T}} \rangle + f\, \mathbf{d}(\langle \boldsymbol{u}, \boldsymbol{\mathcal{T}} \rangle). \tag{11}$$

On the other hand, one can readily verify that

$$\nabla(f\; \boldsymbol{u}) \dot{\wedge} \boldsymbol{\mathcal{T}} = (\boldsymbol{u} \otimes \mathbf{d}f) \dot{\wedge} \boldsymbol{\mathcal{T}} + f\, \nabla \boldsymbol{u} \dot{\wedge} \boldsymbol{\mathcal{T}} = (\mathbf{d}f) \wedge \langle \boldsymbol{u}, \boldsymbol{\mathcal{T}} \rangle + f \nabla \boldsymbol{u} \dot{\wedge} \boldsymbol{\mathcal{T}}, \tag{12}$$

which proves our claim. Note that the differential operator $\mathfrak{d}$ is closely related to *Cartan's exterior covariant differential* (reviewed in [9, Chapter 9], see also [21] and [16]), which was originally developed to express connections and curvatures in terms of forms.

Motivated by the fact that the usual pull-back of forms commutes with the exterior derivative, we are able to use $\mathfrak{d}$ to define a derivative $\mathfrak{D}$ on elements of

[4] Clearly, the $\dot{\wedge}$ operation can be extended to all tensors by linearity.

the space $\mathbb{T} \otimes \Omega^{k-1}(\mathcal{R}_0)$ $(k \leq 3)$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{T} \otimes \Omega^{k-1}(\mathcal{R}_0) & \xleftarrow{\varphi^{*2}} & \mathbb{T} \otimes \Omega^{k-1}(\mathcal{R}) \\ \mathfrak{D} \big\downarrow & & \big\downarrow \mathfrak{d} \\ \mathbb{T} \otimes \Omega^{k}(\mathcal{R}_0) & \xleftarrow[\varphi^{*2}]{} & \mathbb{T} \otimes \Omega^{k}(\mathcal{R}) \end{array}$$

where φ^{*_2} denotes a partial pullback of the "$(k-1)-$form" part of the tensors but does nothing to the vector or covector values. This diagram simply means that for $\nu \in \mathbb{T},\ \omega \in \Omega^{k-1}(\mathcal{R})$, we have

$$\mathfrak{d}\,(\nu \otimes \omega)\,(\boldsymbol{u}, \varphi_* \boldsymbol{V}_1, ..., \varphi_* \boldsymbol{V}_k) = \mathfrak{D}\,(\nu \otimes \varphi^* \omega)\,(\boldsymbol{u}, \boldsymbol{V}_1, ..., \boldsymbol{V}_k), \tag{13}$$

for all $\boldsymbol{u} \in \mathbb{T}^*$ and $\boldsymbol{V}_1, ..., \boldsymbol{V}_k \in T\mathcal{R}_0$. Equation (13) provides a definition for $\mathfrak{D}$.

Useful identities. The following identities hold:[5]

$$(\operatorname{div} \boldsymbol{\sigma}^{\sharp_2}) \otimes \mu \;=\; \mathfrak{d}(*_2 \boldsymbol{\sigma})\ , \qquad\qquad (\operatorname{Div} \mathbf{P}^{\sharp_2}) \otimes \mu_0 \;=\; \mathfrak{D}(*_2 \mathbf{P})\ . \tag{14}$$

where div and Div denote the divergence operator on $\mathcal{R}$ and $\mathcal{R}_0$, respectively, while $\boldsymbol{\sigma}^{\sharp_2} = \sigma_a{}^b \mathbf{e}^a \otimes \mathbf{e}_b$ and $\mathbf{P}^{\sharp_2} = P_A{}^B \mathbf{E}^A \otimes \mathbf{E}_B$, that is, $(.)^{\sharp_2}$ acts on the second 'leg' only. Here, μ and μ_0 are volume forms in $\Omega^3(\mathcal{R})$ and $\Omega^3(\mathcal{R}_0)$, respectively.

2.4. Balance laws and constitutive relations

Balance of linear momemtum. The pointwise equations of balance of linear momentum can be written either with respect to the current configuration $\mathcal{R}$ (Eulerian form) or with respect to the reference configuration $\mathcal{R}_0$ (Lagrangian form) as follows

$$\rho \dot{\mathbf{v}}^\flat = \operatorname{div} \boldsymbol{\sigma}^{\sharp_2} + \rho \mathbf{b}, \qquad\qquad \rho_0 \dot{\mathbf{V}}^\flat = \operatorname{Div} \mathbf{P}^{\sharp_2} + \rho_0 \mathbf{B}, \tag{15}$$

where the overdot denotes the material time derivative. Here, ρ is the mass density in $\mathcal{R}$, while ρ_0 is the referential mass density. Also, one has $\mathbf{B}(\mathbf{X}, t) = \mathbf{b}(\mathbf{x}(\mathbf{X}, t), t)$. It is worth emphasizing here that $\mathbf{V}(\mathbf{X}, t)$, $\mathbf{B}(\mathbf{X}, t)$ and $\mathbf{p}(\mathbf{X}, t)$ are expressed in terms of a base point $\mathbf{X}$ in $\mathcal{R}_0$ but take their values in the tangent (or cotangent) fiber $T_{\mathbf{x}}\mathcal{R}$ (or $T^*_{\mathbf{x}}\mathcal{R}$) above the corresponding point $\mathbf{x} = \varphi(\mathbf{X}, t)$ in $\mathcal{R}$. Take the tensor product of the point-wise balance of linear momentum (15) with the volume forms μ and μ_0, respectively, and use the identities in (14) to get

$$\dot{\mathbf{v}}^\flat \otimes \rho\mu = \mathfrak{d}\boldsymbol{\mathcal{T}} + \boldsymbol{b} \otimes \rho\mu, \qquad\qquad \dot{\mathbf{V}}^\flat \otimes \rho_0\mu_0 = \mathfrak{D}\boldsymbol{\mathcal{P}} + \boldsymbol{B} \otimes \rho_0\mu_0\ . \tag{16}$$

Balance of angular momemtum. Balance of angular momentum states that

$$\boldsymbol{\sigma}^\mathsf{T} = \boldsymbol{\sigma}, \tag{17}$$

[5] One can prove analogous results for any 2-tensor.

Consequently, the tensor $\mathbf{P}\mathbf{F}^{\mathsf{T}} = J\boldsymbol{\sigma}$ is also symmetric. This symmetry translates in terms of $\boldsymbol{\mathcal{T}}$, viewed as a vector-valued two-form, to the following equality:

$$(\nu \otimes \beta) \dot{\wedge} \boldsymbol{\mathcal{T}} = (\beta \otimes \nu) \dot{\wedge} \boldsymbol{\mathcal{T}}, \tag{18}$$

for all $\nu, \beta \in \Omega^1(\mathcal{R})$. The symmetry of $\boldsymbol{\mathcal{T}}$ can be interpreted physically as follows. Consider a surface $S_{\mathbf{n}}$ with unit normal $\mathbf{n}$ moving at a velocity $\mathbf{v_e} = v\,\mathbf{e}$, where $\mathbf{e}$ is a unit vector. From (18), one gets that $(\mathbf{v_e}^\flat \otimes \mathbf{n}^\flat)\dot{\wedge}\boldsymbol{\mathcal{T}} = \mathbf{n}^\flat \wedge \langle \mathbf{v_e}, \boldsymbol{\mathcal{T}} \rangle = \mathbf{e}^\flat \wedge \langle \mathbf{v_n}, \boldsymbol{\mathcal{T}} \rangle$, where $\mathbf{v_n} = v\mathbf{n}$. Combining this with (8), we see that *the power per unit area expended by the stress as a surface $S_{\mathbf{n}}$ with unit normal $\mathbf{n}$ moves at a velocity $\mathbf{v_e} = v\mathbf{e}$ is equal to the power per unit area expended by the stress as a surface $S_{\mathbf{e}}$ with unit normal e moves at a velocity $\mathbf{v_n} = v\mathbf{n}$.*

Finally, note that the statement of balance of angular momentum for $\boldsymbol{\mathcal{P}}$ (as a vector-valued two-form) can be equivalently expressed as

$$(\nu \otimes \varphi^*\beta)\dot{\wedge}\boldsymbol{\mathcal{P}} \;=\; (\beta \otimes \varphi^*\nu)\dot{\wedge}\boldsymbol{\mathcal{P}}. \tag{19}$$

Conservation of mass. Recall that the pointwise equation of conservation of mass is usually written as

$$\rho_0 = \rho J, \tag{20}$$

which can be expressed as $\varphi^*(\rho\mu) = \rho_0\mu_0$.

Balance of energy. For an elastic material one assumes that there exists a strain energy function e per unit mass whose change represents the change in the internal energy due to mechanical deformations. Balance of energy may be written as

$$\int_V \rho\langle \mathbf{v}, \mathbf{b}\rangle\, \mu + \int_S \langle \mathbf{v}, \boldsymbol{\mathcal{T}}\rangle \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_V \rho\, \langle\!\langle \mathbf{v}, \mathbf{v}\rangle\!\rangle\, \mu + \int_V e\,\rho\mu\Big), \tag{21}$$

or, equivalently, on $\mathcal{R}_0$ as

$$\int_{V_0} \rho_0\langle \mathbf{V}, \mathbf{B}\rangle\, \mu_0 + \int_{S_0} \langle \mathbf{V}, \boldsymbol{\mathcal{P}}\rangle \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_{V_0} \rho_0\, \langle\!\langle \mathbf{V}, \mathbf{V}\rangle\!\rangle\, \mu_0 + \int_{V_0} e\,\rho_0\mu_0\Big)\,. \tag{22}$$

Here, the $\langle\!\langle, \rangle\!\rangle$ denotes the inner products both on $T\mathcal{R}$ and $T\mathcal{R}_0$, respectively. The volume integrals are taken over an arbitrary subset $V \subseteq \mathcal{R}$ while $V_0 = \varphi^{-1}(V) \subseteq \mathcal{R}_0$ and the area integrals are taken over the bounding surfaces $S = \partial V$ and $S_0 = \partial V_0$. One can readily check, using Stokes' theorem, the definition of $\mathfrak{d}$ in (10) and the balance of linear momentum (16), that the rate of change of internal energy is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V e\,\rho\mu \;=\; \int_V \nabla\mathbf{v}\;\dot{\wedge}\;\boldsymbol{\mathcal{T}}\;, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0} e\,\rho_0\mu_0 \;=\; \int_{V_0} \nabla\mathbf{V}\;\dot{\wedge}\;\boldsymbol{\mathcal{P}}\;. \tag{23}$$

That is, the *stress power* can be expressed in terms of the stress form as in $\nabla\mathbf{v}\;\dot{\wedge}\;\boldsymbol{\mathcal{T}}$ in $\mathcal{R}$ and $\nabla\mathbf{V}\;\dot{\wedge}\;\boldsymbol{\mathcal{P}}$ in $\mathcal{R}_0$. In the classical theory, the stress power is defined as the inner product $\langle\!\langle \nabla\mathbf{v}, \boldsymbol{\sigma}\rangle\!\rangle$ in $\mathcal{R}$ or $\langle\!\langle \nabla\mathbf{V}, \mathbf{P}\rangle\!\rangle$ in $\mathcal{R}_0$, which can also be written as $\langle\!\langle \dot{\mathbf{F}}, \mathbf{P}\rangle\!\rangle$.

We now present a simple counting argument. In general, $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{P}}$ have nine independent components; balance of linear momentum (16) provides three independent equations of motion, and so does balance of angular momentum (18). Therefore, for a prescribed motion under given body forces, one has a system of nine unknowns and six equations. This means that to obtain a determinate system, one needs to impose constitutive relations on the stress $\boldsymbol{\mathcal{T}}$ (or $\boldsymbol{\mathcal{P}}$) as is done below for the case of hyperelastic materials.

Constitutive equations. The physical behavior of solid bodies depends on their material properties. Mathematically, a specific material, or class of materials, is characterized by a specific functional dependence of the stress tensor on the motion, that is, by a constitutive law. Hyperelastic materials have the property that the internal energy e is function of $\mathbf{F}$ only, that is, $e = e(\mathbf{X}, \mathbf{F})$, and that the dependence of the stress on the motion is given by:

$$\mathbf{P}^{\sharp_1} = \rho_0 \frac{\partial e}{\partial \mathbf{F}} , \qquad \boldsymbol{\sigma} = 2\rho \frac{\partial e}{\partial \boldsymbol{g}} \tag{24}$$

where the second part is known as the Doyle-Ericksen formula (here, $\boldsymbol{g} = g^{ij} \mathbf{e}_i \otimes \mathbf{e}_j$). These relations can be translated to obtain constitutive laws for the new stress by applying the hodge star operator $*_2$, that is,

$$\boldsymbol{\mathcal{P}}^{\sharp_1} = *_2 \left(\rho_0 \frac{\partial e}{\partial \mathbf{F}} \right) , \qquad \boldsymbol{\mathcal{T}} = *_2 \left(2\rho \frac{\partial e}{\partial \boldsymbol{g}} \right) , \tag{25}$$

It is worth noting that in (21-24), one could consider the strain energy function as an energy density or a volume form by treating $e\rho\mu$ as a single object, say $\epsilon \in \Omega^3(\mathcal{R})$ and $e\rho_0\mu_0$ as a single object $\epsilon_0 \in T^0_0(\mathcal{R}) \otimes \Omega^3(\mathcal{R}_0)$.[6] The representation of the stress as a covector-valued two-form is consistent with the stress object one obtains from (24) using the energy density. For example, one can readily verify that $\boldsymbol{\mathcal{P}} = \text{trace}(\partial\epsilon_0/\partial\mathbf{F})$.

Remark. The reader is reminded that the existence of the stress tensors (3) and the pointwise dynamical equations (15), (17) and (20) can be obtained by postulating integral laws of balance of momenta for the body, which is assumed to deform in the Euclidean space, or by postulating a *covariant*[7] balance of energy for the body deforming in a general Riemannian manifold, see, [17] and [23]. It is important to note that the integral laws of balance of momenta are not intrinsic; the notion of a resultant force (or moment) explicitly utilizes the fact that the underlying physical space is Euclidean in a way that cannot be generalizable to curved Riemannian manifolds. In §3 we outline a procedure for deriving the pointwise balance laws from a fully geometric covariant theory of elasticity.

[6] This notion of energy density is consistent with what is done in classical field theory, for example, in electromagnetism, the Lagrangian is not a scalar valued function but a density.
[7] Here, by *covariant* we mean invariant under general coordinate transformations.

3. Covariant derivation of the balance laws

In covariant elasticity, one starts from the balance of energy in (21) and postulates that it is invariant under arbitrary spatial diffeomorphisms $\xi_t : \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{S}}$, in order to derive the local forms of the conservation of mass, balance of momenta and the Doyle-Ericksen formula, see [17] and [23]. This covariant approach is a beautiful generalization of the classical Green-Rivlin-Naghdi results on invariance of energy balance under rotations and translations in Euclidean space. The main addition that one needs to make this covariant under *general transformations* are the stress constitutive relations.

In what follows, we assume the existence of a stress form $\boldsymbol{\mathcal{T}}$, with no reference to the stress tensor $\boldsymbol{\sigma}$, and present a covariant derivation of the balance laws and constitutive equations. For notational convenience, define a scalar $f := e + \frac{1}{2}\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle$; one has

$$\frac{d}{dt}\int_V f\rho\mu \;=\; \int_V \mathbf{L_v}\left(f\rho\mu\right) \;=\; \int_V \rho\mu\mathbf{L_v} f + f\mathbf{L_v}(\rho\mu)\ , \tag{26}$$

where $\mathbf{L_v}$ denotes the Lie derivative with respect to the velocity field $\mathbf{v}$ and $\mathbf{L_v}f = \dot{e} + \langle\!\langle \mathbf{v}, \dot{\mathbf{v}} \rangle\!\rangle$. Given a spatial diffeomorphism ξ_t, the balance of energy (21) can be rewritten in $\boldsymbol{\mathcal{S}}$ as follows:

$$\frac{d}{dt}\int_{V'} \big(e' + \frac{1}{2}\langle\!\langle \mathbf{v}', \mathbf{v}' \rangle\!\rangle\big)\rho'\mu' = \int_{V'} \rho'\langle \mathbf{b}', \mathbf{v}'\rangle\mu' + \int_{S'} \left\langle \boldsymbol{\mathcal{T}}', \mathbf{v}' \right\rangle\ . \tag{27}$$

where the prime notation $()'$ is used to denote quantities in $\xi_t(\boldsymbol{\mathcal{S}})$. In particular, one has $\rho\mu = \xi_t^*(\rho'\mu')$ (conservation of mass) and $\mathbf{v}' = \xi_{t*}\mathbf{v} + \mathbf{w}_t$, where $\mathbf{w}_t$ is the velocity of ξ_t. Further, one considers that the body force $\mathbf{b}$ transforms under spatial diffeomorphisms according to $\mathbf{b} - \dot{\mathbf{v}}^\flat = \xi_t^*\left(\mathbf{b}' - \dot{\mathbf{v}}'^\flat\right)$, also, one has that the internal energy depends parametrically on the metric $e'(\mathbf{x}', t, \mathbf{g}) = e(\mathbf{x}, t, \xi_t^*\mathbf{g})$, (See [17, Chapter 2, Box 3.1].) Now, similarly to (26), one has

$$\frac{d}{dt}\int_{V'} f'\rho'\mu' \;=\; \int_{V'} \rho'\mu'\mathbf{L}_{\mathbf{v}'}f' + f'\mathbf{L}_{\mathbf{v}'}(\rho'\mu')\ , \tag{28}$$

where $f' = e' + \frac{1}{2}\langle\!\langle \mathbf{v}', \mathbf{v}' \rangle\!\rangle$, $\mathbf{L}_{\mathbf{v}'}(\rho'\mu') = \xi_{t*}\left(\mathbf{L_v}(\rho\mu)\right)$ and $\mathbf{L}_{\mathbf{v}'}f' = \dot{e}' + \langle\!\langle \mathbf{v}', \dot{\mathbf{v}}' \rangle\!\rangle$. To this end, at $t = t_0$, one gets that $f'|_{t=t_0} = f + \langle\!\langle \mathbf{v}, \mathbf{w} \rangle\!\rangle + \frac{1}{2}\langle\!\langle \mathbf{w}, \mathbf{w} \rangle\!\rangle$, as well as $\mathbf{L}_{\mathbf{v}'}(\rho'\mu')|_{t=t_0} = \mathbf{L_v}(\rho\mu)$, and

$$\left(\mathbf{L}_{\mathbf{v}'}f'\right)\big|_{t=t_0} \;=\; \dot{e} + \left\langle\!\!\left\langle \frac{\partial e}{\partial \mathbf{g}}, \mathbf{L_w}\mathbf{g} \right\rangle\!\!\right\rangle + \langle\!\langle \mathbf{v} + \mathbf{w}, \dot{\mathbf{v}}' \rangle\!\rangle\ . \tag{29}$$

Substitute (28-29) into (27) and subtract from (21) (in the resulting equation, the only term with $\langle\!\langle \mathbf{w}, \mathbf{w} \rangle\!\rangle$ is $\langle\!\langle \mathbf{w}, \mathbf{w} \rangle\!\rangle\mathbf{L_v}(\rho\mu)$, hence, the arbitrariness of $\mathbf{w}$ leads to $\mathbf{L_v}(\rho\mu) = 0$, *i.e.*, mass conservation). Take mass conservation into consideration and simplify the resulting equation to get

$$\int_V \left\langle\!\!\left\langle \frac{\partial e}{\partial \mathbf{g}}, \mathbf{L_w}\mathbf{g} \right\rangle\!\!\right\rangle \rho\mu \;=\; \int_V \langle \mathbf{w}, \mathbf{b} - \dot{\mathbf{v}}^\flat\rangle\rho\mu + \int_S \langle \mathbf{w}, \boldsymbol{\mathcal{T}}\rangle\ . \tag{30}$$

Apply Stokes' theorem to the last term and appeal to (10) to get

$$\int_V \left\langle\!\!\left\langle \frac{\partial e}{\partial \mathbf{g}}, \mathbf{L_w g} \right\rangle\!\!\right\rangle \rho\mu \;=\; \int_V \langle \mathbf{w}, \mathbf{b} - \dot{\mathbf{v}}^\flat \rangle \rho\mu + \int_V \langle \mathbf{w}, \mathfrak{d}\boldsymbol{\mathcal{T}} \rangle + \int_V \nabla \mathbf{w} \,\dot{\wedge}\, \boldsymbol{\mathcal{T}}. \qquad (31)$$

Now $\nabla\mathbf{w}$ can be written as the sum of its symmetric and skew-symmetric parts, namely as

$$(\nabla \mathbf{w})^\flat = \frac{1}{2}\mathbf{L_w g} + \mathrm{d}(\mathbf{w}^\flat)$$

(it is important to note here that $\mathbf{g}$ is time independent). Because at any point, $\mathbf{w}$, $\mathbf{L_w g}$, and $\mathrm{d}(\mathbf{w}^\flat)$ can be chosen independently, one gets the balance of linear momentum (16), balance of angular momentum (18), and Doyle-Ericksen formula $(25)_2$.

4. Summary

This paper has presented a new and geometrically more natural formulation of continuum mechanics in terms of vector- and covector-valued forms, which are taken as replacements for (and are equivalent to) the standard stresses. It was shown that the formulation is equivalent to the classic theory by introducing mathematical operations on the relevant tangent and cotangent bundles. In this reformulation, the Cauchy stress field is replaced by a covector-valued two-form $\boldsymbol{\mathcal{T}}$ which, when paired with the velocity field and integrated over a surface S, gives the rate of work done by the stress on that surface. Cauchy's Lemma is also automatically satisfied from the geometric nature of the stress $\boldsymbol{\mathcal{T}}$. Finally, we presented a covariant derivation of the balance laws and constitutive relations in terms of $\boldsymbol{\mathcal{T}}$ directly without the need to utilize the classical notion of stress as a two-tensor. As future work, it is planned to derive a discrete implementation of this geometric standpoint and compare it to recent numerical techniques [12, 7] that seem to share similar ideas and properties.

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